Quantum and Superquantum Nonlocal Correlations*

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We present a simple hidden variable model for the singlet state of a pair of qubits, characterized by two kinds, hierarchically ordered, of hidden variables. We prove that, averaging over both types of variables, one reproduces all the quantum mechanical correlations of the singlet state. On the other hand, averaging only over the hidden variables of the lower level, one obtains a general formal theoretical scheme exhibiting correlations stronger than the quantum ones, but with faster-than-light communication forbidden. This result is interesting by itself since it shows that a violation of the quantum bound for nonlocal correlations can be implemented in a precise physical manner and not only mathematically, and it suggests that resorting to two levels of nonlocal hidden variables might led to a deeper understanding of the physical principles at the basis of quantum nonlocality.

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Introduction — Nonlocal correlations have always been considered as one of the peculiar traits of quantum mechanics. Their relevant conceptual implications have been already recognized by the founders of the quantum theory, as clearly expressed by Schrödinger [1]:

> *It is rather discomforting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter's mercy in spite of his having no access to it.*

Their potential impact for technological implementations is at the basis of the theories of quantum information, computation and cryptography. Nonetheless, the physical principles underlying quantum nonlocality are rather obscure (see [2] for recent developments).

One of the most relevant features of quantum correlations is that, while they can produce instantaneous actions at a distance [3], they cannot be used for faster-than-light communication. This no-signalling property allows quantum mechanics to peacefully coexist with relativistic causality. Nevertheless, such a property is also consistent with generalized models which are more nonlocal than quantum mechanics [4]. Accordingly, one cannot account for the specific form of quantum nonlocality in terms of causality requirements. The investigation of the superquantum correlations generated by these models is important from an information theoretic point of view, and it is expected to lead to a deeper understanding of the physical principles at the basis of quantum mechanics [5–7]. In this perspective, any connection between generalized no-signalling models and quantum mechanics might represent a relevant conceptual step for the understanding of the origin and of the role of nonlocality in the quantum domain.

Recently, Leggett exhibited a no-signalling hidden variable model of a new type [8], which stimulated investigations on the nature of quantum nonlocality [9–12]. In this paper we present a simple hidden variable model for the singlet of a two-qubits system, which is strongly related to the model described by Bell in his analysis of nonlocality [13], and constitutes an example of the theories considered by Leggett. Although it is just a toy model, it throws an interesting light on the general problem of nonlocality, since it provides a meaningful connection between the main ideas of [4, 8]. In fact, it can produce nonquantum nonlocal correlations, which reduce to the the standard quantum ones when integration over all hidden variables is performed. This emergence of quantum correlations from a generalized no-signalling model was partially obtained in a former work [7], without reference to hidden variables theories.

In the following we consider a bipartite system shared between two spatially separate observers, which can independently choose inputs $\mathbf{a}$ and $\mathbf{b}$, and obtain outputs a and b. In general, $\mathbf{a}$ and $\mathbf{b}$ are the physical setting which separately determine the local observables A and B, and a, b are the respective measurement results.
We start by briefly reconsidering the arguments of [4, 8].

The analysis of Popescu and Rohlrich — To illustrate the arguments of these authors we follow their line of thought. Let A, A', B and B' be physical variables taking values $+1$ and -1, with A and A' referring to measurements on one part of the system and B and B' referring to the other part. If we denote as $P_{AB}(a, b)$ the joint probability of obtaining $A = a$ and $B = b$ when both A and B are measured, the correlation $E(A, B)$ of the outcomes is defined as:

$$\begin{aligned} E(A,B) &= P_{AB}(+1,+1) + P_{AB}(-1,-1) + \\ &- P_{AB}(+1,-1) - P_{AB}(-1,+1). \end{aligned} \tag{1}$$

As well known Clauser, Horne, Shimony and Holt [14]

have shown, completely in general, that an appropriate combination of correlations involving four arbitrary directions satisfy, for all local theories, the inequality $|F| \leq 2$, where

$$F \equiv E(A,B) + E(A,B') + E(A',B) - E(A',B'). \quad (2)$$

On the other hand, in the quantum case, when the system is in an entangled pure state, the correlations $E_\psi(A,B) \equiv \langle\psi|A \otimes B|\psi\rangle$ violate, for appropriate choices of the observables, the above inequality. Actually, Bell [13] has derived his celebrated inequality, $|F_\psi| \leqslant 2\sqrt{2}$, where F_ψ has the same form of (2) with $E_\psi(X,Y)$ replacing $E(X,Y)$. It has to be noted that the value $2\sqrt{2}$ represents the maximal possible violation of locality which can occur within quantum mechanics.

The authors of [4] have investigated whether the request that the hypothetical general nonlocal theory one is envisaging respects relativistic causality might be responsible for the precise value of Bell's upper bound. The question is interesting since, at first sight, one might expect that the above combination of correlations reaches the value 4, which is attained when the first three terms take the value $+1$ and the last the value -1 (or viceversa). The extremely interesting result of [4] is that a nonlocal theory respecting relativistic causality and yielding a violation of Bell's bound is actually possible. We will call any theory exhibiting such a feature a superquantum nonlocal theory. Therefore, Bell's bound does not follow from a fundamental principle (preservation of causality), but it is rather a consequence of the structure of quantum mechanics.

The prototype of a device which is more nonlocal than quantum mechanics, still consistent with relativistic causality, is the so-called *Popescu-Rohrlich (PR) box* [5, 6]. Its inputs, denoted here by $\mathbf{a}$ and $\mathbf{b}$ for consistency, independently chosen by Alice and Bob respectively, are classical bits which assume the values $\{0,+1\}$. Its outputs, denoted by a and b respectively, are classical bits taking the same values, determined by the relation $\mathbf{ab} = a + b$, where the sum is modulo 2. Alice inputs $\mathbf{a}$ and extracts a, and Bob inputs $\mathbf{b}$ and extracts b. If the outputs are redefined as $a' = 1 - 2a$, $b' = 1 - 2b$, such that the outcomes are $\{-1,+1\}$, it is possible to check that the PR box provides correlations satisfying $|F| = 4$. To our knowledge, the only possible implementation of the PR box so far discussed requires post-measurement selection [15].

Leggett's proposal — Leggett embraces the perspective that quantum mechanics is an incomplete theory, and argues that the hidden variables required to complete it could enable a deeper understanding of quantum correlations. The hidden variables fully specify the values assumed by the physical quantities in a quantum state ψ, denoted by a_ψ and b_ψ; moreover, Leggett assumes that the violation of locality does respect parameter independence, i.e. that while the value of $a_\psi(\mathbf{a},\mathbf{b},\lambda)$ might depend on both settings $\mathbf{a}$ and $\mathbf{b}$, it does not depend on the value taken by B [16]. As usual, the ignorance on the specific values of the hidden variables is responsible for the statistical content of quantum mechanics. It is a well known fact that all single and joint outcomes implied by a maximally entangled state cannot be described in terms of a classical distribution over the set of hidden variables. Nevertheless, Leggett has investigated wether it would be possible - at least in principle - to have a local description of local outcomes when only a subset of the hidden variables is taken into account to evaluate the averages. He has characterized his attempt as taking into account *crypto nonlocal theories*.

The ideas of Leggett, originally introduced in the description of the rotationally invariant maximally entangled state of the polarization degrees of freedom of a pair of photons, are summarized here in terms of a more general theory. The hidden variables are given by $\lambda = (\mu,\tau)$, where μ and τ are two families of hidden variables. It is assumed that

$$\begin{aligned} \langle A \otimes B\rangle_\psi &= \int E_{\psi,\tau}(A,B)\,\rho(\tau)\,d\tau, \\ \langle A\rangle_\psi &= \int f_\psi(\mathbf{a},\tau)\,\rho(\tau)\,d\tau, \end{aligned} \quad (3)$$

and similarly for $\langle B\rangle_\psi$, with $g_\psi(\mathbf{b},\tau)$ taking the place of $f_\psi(\mathbf{b},\tau)$. In this equation, the intermediate averages over μ are given by

$$\begin{aligned} E_{\psi,\tau}(A,B) &= \int a_\psi(\mathbf{a},\mathbf{b},\mu,\tau)b_\psi(\mathbf{a},\mathbf{b},\mu,\tau)\,\rho(\mu|\tau)\,d\mu, \\ f_\psi(\mathbf{a},\tau) &= \int a_\psi(\mathbf{a},\mathbf{b},\mu,\tau)\,\rho(\mu|\tau)\,d\mu, \\ g_\psi(\mathbf{b},\tau) &= \int b_\psi(\mathbf{a},\mathbf{b},\mu,\tau)\,\rho(\mu|\tau)\,d\mu, \end{aligned} \quad (4)$$

The constraints on local measurements, embodied by the second and third of (4), provide the fundamental condition imposed to the model, denoted by Leggett as *crypto-nonlocality*, i.e., that averaging the single particles values for A and B over the "deep level" hidden variables μ, nonlocality is washed out. This is a no-signalling condition.

Leggett himself has proved that his proposal conflicts with some predictions of quantum mechanics. Moreover, his work has stimulated a series of interesting investigations [9–12], proving in particular that any crypto-nonlocal theory equivalent to quantum mechanics must be characterized by $f(\mathbf{a},\tau) = 0$ and $g(\mathbf{b},\tau) = 0$.

These conclusions might be (and actually have been) taken as a proof that it is useless to consider crypto-nonlocal theories (i.e. theories with two levels of the hidden variables) since they would, in practice, turn out to be a sort of local theories which are already known to be incompatible with quantum predictions. However, as we are going to prove, this is not the case, because it is possible to work out theories of this type which reproduce

the quantum single particle expectation values when integrated on the deeper level hidden variables, but which require also the integration on the upper level variables in order to reproduce the quantum correlations, while, at the lower level, they violate the quantum upper bound of $2\sqrt{2}$ on the combination of the correlations. In brief, at their lower level such theories are superquantum nonlocal.

Our proposal — We take inspiration by the simple nonlocal hidden variable model introduced by Bell in his celebrated paper. However, we improve it by making it more symmetric for what concerns its nonlocal features, and we modify it in order to make it of the crypto-nonlocal type. The model describes a pair of qubits in the singlet state ψ in terms of a hidden variable, which is a unit vector λ in three dimensional space. In this space we fix an orthogonal reference frame, denoted as (x, y, z). The vector λ is assumed to be uniformly distributed over the unit sphere, and will be uniquely specified by polar angles μ and τ which we unconventionally choose to take values in the intervals $\mu \in [0, 2\pi)$, $\tau \in [0, \pi)$. Such variables are related to the standard polar angles θ and ϕ according to [17]:

$$\begin{aligned} \mu = \theta,\ \tau = \phi \qquad & \text{for } y \geqslant 0; \\ \mu = 2\pi - \theta,\ \tau = \phi - \pi \qquad & \text{for } y < 0. \end{aligned} \tag{5}$$

The assignment of $\lambda = (\mu, \tau)$ uniquely determines the nonlocally possessed values (the certain measurement outcomes) of $A = \mathbf{a} \cdot \sigma$ and $B = \mathbf{b} \cdot \sigma$ according to:

$$\begin{aligned} a_\psi(\mathbf{a}, \mathbf{b}, \mu, \tau) &= \operatorname{sign}(\hat{\mathbf{a}} \cdot \lambda), \\ b_\psi(\mathbf{a}, \mathbf{b}, \mu, \tau) &= -\operatorname{sign}(\hat{\mathbf{b}} \cdot \lambda). \end{aligned} \tag{6}$$

The vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ lie in the plane identified by $\mathbf{a}$ and $\mathbf{b}$, and are obtained from these vectors by rotating them in such a way that they are still symmetrically disposed with respect to the bisector of the angle ω (with $0 \leqslant \omega \leqslant \pi$) between $\mathbf{a}$ and $\mathbf{b}$, and $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ form an angle $\hat{\omega}$ satisfying, as in the case of Bell's model, $\hat{\omega} = \pi \sin^2 \frac{\omega}{2}$. Notice that $\hat{\omega} \leqslant \omega$ when $\omega \leqslant \pi/2$, and $\hat{\omega} > \omega$ when $\omega > \pi/2$.

It turns out that our model is crypto-nonlocal, with the variables μ and τ playing the role of lower and upper level hidden variables, respectively. In fact, from (6) it follows that every observable assumes the values $+1$ and -1 in opposite hemispheres of the unit sphere of λ, which has uniform distribution. Integration over μ means integration over a maximal circle, therefore $f_\psi(\mathbf{a}, \tau) = g_\psi(\mathbf{b}, \tau) = 0$, which, as already stressed, is a necessary condition for crypto-nonlocality, in the case of the singlet.

To proceed, we evaluate now the averages on the variable μ of the correlation functions, i.e., expressions of the type:

$$E_{\psi,\tau}(A, B) = \frac{1}{4} \int_0^{2\pi} a_\psi(\mathbf{a}, \mathbf{b}, \mu, \tau) b_\psi(\mathbf{a}, \mathbf{b}, \mu, \tau) |\sin \mu| d\mu. \tag{7}$$

For simplicity, we will limit our consideration to 4 directions in the (x, z)-plane such that:

$$\begin{aligned} \mathbf{a} = (\sin\alpha, 0, \cos\alpha),\ \ \mathbf{a}' = (-\sin 3\alpha, 0, \cos 3\alpha), \\ \mathbf{b} = (-\sin\alpha, 0, \cos\alpha),\ \ \mathbf{b}' = (\sin 3\alpha, 0, \cos 3\alpha), \end{aligned} \tag{8}$$

determining the dichotomic observables A, B, A' and B', as usual. To exhibit the emergence of quantum nonlocality in the standard scenario, and also of superquantum nonlocality, in our context, it is sufficient to limit the attention to the interval $\alpha \in [0, \pi/4]$. Let $\tilde{\alpha}$ be the solution of $4\alpha + \widehat{2\alpha} = \pi$, where $\widehat{2\alpha} = \pi \sin^2 \alpha$ (it turns out that $\tilde{\alpha} \simeq 0.316$). For the aforementioned choice of physical settings, we obtain $E_{\psi,\tau}(A, B) = 2|\chi_1| - 1$, $E_{\psi,\tau}(A', B') = 2|\chi_2| - 1$,

$$E_{\psi,\tau}(A, B') = E_{\psi,\tau}(A', B) = |\chi_3 - \chi_4| - 1 \tag{9}$$

when $0 \leqslant \alpha \leqslant \tilde{\alpha}$, and

$$E_{\psi,\tau}(A, B') = E_{\psi,\tau}(A', B) = 1 - |\chi_3 + \chi_4|$$

when $\tilde{\alpha} < \alpha \leqslant \pi/4$. In the above equations we have defined

$$\chi_j = \chi_j(\alpha, \tau) = \frac{\cos\tau}{\sqrt{\cos^2\tau + \cot^2 \frac{\gamma_j(\alpha)}{2}}}, \tag{10}$$

with the functions $\gamma_j(\alpha)$ given by:

$$\begin{aligned} \gamma_1(\alpha) = \pi \sin^2 \alpha, \quad \gamma_2(\alpha) = \pi \sin^2 3\alpha, \\ \gamma_3(\alpha) = 4\alpha + \pi \sin^2 \alpha, \quad \gamma_4(\alpha) = 4\alpha - \pi \sin^2 \alpha. \end{aligned} \tag{11}$$

We briefly describe how to derive the simplest joint correlation, $E_{\psi,\tau}(A, B)$. By using that

$$\begin{aligned} \operatorname{sign}(\hat{\mathbf{a}} \cdot \lambda) \operatorname{sign}(\hat{\mathbf{b}} \cdot \lambda) &= \operatorname{sign}(\hat{\mathbf{a}} \cdot \lambda)(\hat{\mathbf{b}} \cdot \lambda) \\ &= \operatorname{sign}(\chi_1^2 - \cos^2 \mu), \end{aligned} \tag{12}$$

we compute

$$E_{\psi,\tau}(A, B) = \frac{1}{2} \int_0^\pi \operatorname{sign}(\chi_1^2 - \cos^2 \mu) \sin \mu \, d\mu, \tag{13}$$

from which the result follows. The other cases are rather cumbersome, but follow the same line of thought.
We combine now the joint correlations as usual, i.e., as in (2), with $E_{\psi,\tau}(X, Y)$ replacing $E(X, Y)$, getting

$$F_{\psi,\tau} = \begin{cases} 2(|\chi_1| - |\chi_2| + |\chi_3 - \chi_4| - 1), & \alpha \in [0, \tilde{\alpha}], \\ \\ 2(|\chi_1| - |\chi_2| - |\chi_3 + \chi_4| + 1), & \alpha \in [\tilde{\alpha}, \frac{\pi}{4}]. \end{cases} \tag{14}$$

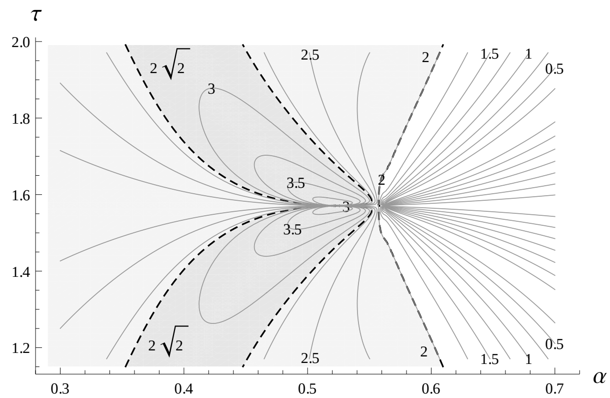

FIG. 1: Contour plot of the function $|F_{\psi,\tau}|$, in the space of the parameters (α, τ). This space is divided in three regions, corresponding to locality, $|F_{\psi,\tau}| \leqslant 2$, quantum nonlocality, $2 < |F_{\psi,\tau}| \leqslant 2\sqrt{2}$, and superquantum nonlocality, $2\sqrt{2} < |F_{\psi,\tau}| \leqslant 4$, identified by the dashed lines.

The function $F_{\psi,\tau}$ takes in general both positive and negative values lying in the interval $[-4, 4]$. However, it is easy to check that, averaging this function over the remaining hidden variable τ, we obtain the quantum mechanical expression F_ψ. In fact,

$$\frac{1}{\pi}\int_0^\pi \chi_j(\alpha,\tau)\,d\tau = \frac{2}{\pi}\,\gamma_j(\alpha), \tag{15}$$

producing $F_\psi = -\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b}'-\mathbf{a}'\cdot\mathbf{b}+\mathbf{a}'\cdot\mathbf{b}'$, which, as well known, saturates the quantum limit $2\sqrt{2}$ at $\alpha = \pi/8$.

Let us come back to study the function $|F_{\psi,\tau}|$, i.e. the modulus of the standard combination of values when the average is taken only on the lower level variables. This function is continuous, exception made for a singular point at $(\alpha, \tau) = (\pi/6, \pi/2)$; in the neighborhoods of this point the upper bound 4 is approached,

$$\sup |F_{\psi,\tau}| = 4. \tag{16}$$

Summarizing, the correlations of the simple model we have presented, when only the averages over the lower level variables are taken into account, approach, for values of the variable τ in an appropriate interval, the maximal possible violation of locality in a two-qubits system. Therefore, our model, which is physically meaningful, as shown by its similarity with standard hidden variables models, is not formulated purely in formal mathematical terms and still it is able to simulate the Popescu-Rohrlich box from the point of view of its violation of locality. We can partition the space of parameters (α, τ) in three regions, corresponding to locality, quantum nonlocality, and superquantum nonlocality, see Fig. 1.

Conclusions — We have developed a simple nonlocal model for the singlet state of the two-qubits system, which is predictively equivalent to quantum mechanics when one integrates over all the hidden variables, while exhibiting superquantum nonlocality at the level of the lower hidden variables. This model is strongly based on the famous model of J. Bell, and differs from it only by the way in which nonlocality enters the description, and by the fact that a partial integration over the hidden variables already erases nonlocality, when dealing with local observables.

The main finding of this paper is that nonlocal hidden variables theories of this type, at the intermediate level can exhibit the largest possible nonlocality, in a bipartite system of two qubits. In our model, the upper bound 4 is approached only asymptotically, but we believe that models in which this value is effectively attained are possible.

As a consequence of our analysis, the PR device, and more generally any device exhibiting superquantum correlations, is not only a mathematical tool. There is indeed a strong connection between hidden variables theories and superquantum correlations. In their inspiring work, Rohrlich and Popescu were looking for a physical principle, relativistic causality, which could justify the precise way nonlocality enters the quantum mechanical description. They concluded that this was not the case. Our analysis suggests that this idea can be revitalized in the domain of crypto-nonlocal models, which deserve a particular attention in the identification of the physical principles underlying nonlocality.

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[17] Note that, with this functional change, the surface element of the sphere turns out to be $d\Omega = |\sin\mu| d\mu d\tau$